8. Two-way crossed classifications

Example 8.1 Days to germination of three varieties of carrot seed grown in two types of potting soil.

Soil Type	Variety 1	Variety 2	Variety 3
1	$Y_{111} = 6$ $Y_{112} = 10$ $Y_{113} = 11$	$Y_{121} = 13$ $Y_{122} = 15$	$Y_{131} = 14$ $Y_{132} = 22$
2	$Y_{211} = 12$ $Y_{212} = 15$ $Y_{213} = 19$ $Y_{214} = 18$	$Y_{221} = 31$	$Y_{231} = 18$ $Y_{232} = 9$ $Y_{233} = 12$

This is called an "unbalanced" factorial experiment.

Sample sizes

Soil type	Variety 1	Variety 2	Variety 3
1	$n_{11} = 3$	$n_{12} = 2$	$n_{13} = 2$
2	$n_{21} = 4$	$n_{22} = 1$	$n_{23} = 3$

In general we have

$i = 1, 2, \ldots, a$	levels for the first factor
$j = 1, 2, \ldots, b$	levels for the second factor
$n_{ij} > 0$	observations at the i-th level of the first factor and the j-th level of the second factor

We will restrict our attention to normal-theory Gauss-Markov models.

"Cell means" model:

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2) \quad \begin{cases} i = 1, \ldots, a \\ j = 1, \ldots, b \\ k = 1, \ldots, n_{ij} \end{cases}$$

Clearly, $E(Y_{ijk}) = \mu_{ij}$ is estimable if $n_{ij} > 0$.

Overall mean response:

$$\bar{\mu}_{..} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}$$

Mean response at the i-th level of factor 1, averaging across the levels of factor 2.

$$\bar{\mu}_{i.} = \frac{1}{b} \sum_{j=1}^{b} \mu_{ij}$$

Mean response at the j-th level of factor 2, averaging across the levels of factor 1

$$\bar{\mu}_{.j} = \frac{1}{a} \sum_{i=1}^{a} \mu_{ij}$$

Contrasts of interest

"main effects" for factor 1:

$\bar{\mu}_{i.} - \bar{\mu}_{..} \qquad i = 1, 2, \ldots, a$

$\bar{\mu}_{i.} - \bar{\mu}_{k.} \qquad i \neq k$

"main effects" for factor 2:

$\bar{\mu}_{.j} - \bar{\mu}_{..} \qquad j = 1, 2, \ldots, b$

$\bar{\mu}_{.j} - \bar{\mu}_{.\ell} \qquad j \neq \ell$

Conditional Effects

$$\mu_{ij} - \mu_{kj} \quad \begin{cases} i \neq k \\ j = 1, 2, \ldots, b \end{cases}$$

$$\mu_{ij} - \mu_{i\ell} \quad \begin{cases} j \neq \ell \\ i = 1, 2, \ldots, a \end{cases}$$

Interaction Contrasts

$$\begin{aligned} &(\mu_{ij} - \mu_{kj}) - (\mu_{i\ell} - \mu_{k\ell}) \\ &\quad = (\mu_{ij} - \mu_{i\ell}) - (\mu_{kj} - \mu_{k\ell}) \\ &\quad = \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} \end{aligned}$$

All of these contrasts are estimable when

$$n_{ij} > 0 \qquad \text{for all } (i, j)$$

because

- $E(\bar{Y}_{ij.}) = \mu_{ij}$
- Any linear function of estimable functions is estimable

An "effects" model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2)$$

$$\begin{aligned} i &= 1, 2, \ldots, a \\ j &= 1, 2, \ldots, b \\ k &= 1, 2, \ldots, n_{ij} > 0 \end{aligned}$$

$$
\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{113} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{213} \\ \epsilon_{214} \\ \epsilon_{221} \\ \epsilon_{231} \\ \epsilon_{232} \\ \epsilon_{233} \end{bmatrix}
$$

The resulting restricted model is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2) \quad \begin{cases} i = 1, \dots, a \\ j = 1, \dots, b \\ k = 1, \dots, n_{ij} \end{cases}$$

and

$$\begin{aligned} \alpha_1 &= 0 \\ \beta_1 &= 0 \\ \gamma_{i1} &= 0 \quad \text{for all} \quad i = 1, \dots, a \\ \gamma_{1j} &= 0 \quad \text{for all} \quad j = 1, \dots, b \end{aligned}$$

We will call these "baseline" restrictions.

Soil	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu$	$\mu_{12} = \mu + \beta_2$	$\mu_{13} = \mu + \beta_3$	$\mu + \frac{\beta_1+\beta_2}{3}$
2	$\mu_{21} = \mu + \alpha_1$	$\mu_{22} = \mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\mu_{23} = \mu + \alpha_2 + \beta_3 + \gamma_{23}$	$\mu + \alpha_2 + \frac{\beta_2+\beta_3}{3} + \frac{\gamma_{22}+\gamma_{23}}{3}$
	$\mu + \frac{\alpha_2}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_2 + \frac{\gamma_{22}}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_3 + \frac{\gamma_{23}}{2}$	

Interpretation:

$$\mu = \mu_{11} = E(Y_{11k})$$

is the mean response with both factors at the first level.

$$\alpha_i = \mu_{i1} - \mu_{11} = E(Y_{i1k}) - E(Y_{11k})$$

is the difference in mean responses between levels i and 1 of factor 1 when factor 2 is at level 1.

Soil	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu$	$\mu_{12} = \mu + \beta_2$	$\mu_{13} = \mu + \beta_3$	$\mu + \frac{\beta_1+\beta_2}{3}$
2	$\mu_{21} = \mu + \alpha_1$	$\mu_{22} = \mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\mu_{23} = \mu + \alpha_2 + \beta_3 + \gamma_{23}$	$\mu + \alpha_2 + \frac{\beta_2+\beta_3}{3} + \frac{\gamma_{22}+\gamma_{23}}{3}$
	$\mu + \frac{\alpha_2}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_2 + \frac{\gamma_{22}}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_3 + \frac{\gamma_{23}}{2}$	

Interpretation

$$\beta_j = \mu_{1j} - \mu_{11} = E(Y_{1jk}) - E(Y_{11k}) \quad \text{for } j = 1, 2, \dots, b$$

is the difference in the mean responses for levels j and 1 of factor 2 when factor 1 is at level 1.

Soil	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu$	$\mu_{12} = \mu + \beta_2$	$\mu_{13} = \mu + \beta_3$	$\mu + \frac{\beta_1+\beta_2}{3}$
2	$\mu_{21} = \mu + \alpha_1$	$\mu_{22} = \mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\mu_{23} = \mu + \alpha_2 + \beta_3 + \gamma_{23}$	$\mu + \alpha_2 + \frac{\beta_2+\beta_3}{3} + \frac{\gamma_{22}+\gamma_{23}}{3}$
	$\mu + \frac{\alpha_2}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_2 + \frac{\gamma_{22}}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_3 + \frac{\gamma_{23}}{2}$	

Interaction:

$$\begin{aligned} \gamma_{ij} &= (\mu_{ij} - \mu_{ib}) - (\mu_{aj} - \mu_{ab}) \\ &= (\mu_{ij} - \mu_{aj}) - (\mu_{ib} - \mu_{ab}) \end{aligned}$$

Note that

$$\gamma_{ij} - \gamma_{i\ell} - \gamma_{kj} + \gamma_{k\ell} = \mu_{ij} - \mu_{i\ell} - \mu_{kj} + \mu_{k\ell}$$

for any (i, j) **and** (k, ℓ)

Matrix formulation:

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_2 \\ \beta_2 \\ \beta_3 \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{113} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{213} \\ \epsilon_{214} \\ \epsilon_{221} \\ \epsilon_{231} \\ \epsilon_{232} \\ \epsilon_{233} \end{bmatrix}$$

$$\uparrow \qquad \uparrow$$

$$\mathrm{Y} = X\beta + \epsilon$$

where

$$\mathrm{Y} \sim N(X\beta, \sigma^2 I)$$

Least squares estimation:

$$\begin{aligned} \mathrm{b} &= \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_2 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\gamma}_{22} \\ \hat{\gamma}_{23} \end{bmatrix} \\ &= (X^T X)^{-1} X^T \mathrm{Y} \\ &= \begin{bmatrix} \bar{Y}_{11\bullet} \\ \bar{Y}_{21\bullet} - \bar{Y}_{11\bullet} \\ \bar{Y}_{12\bullet} - \bar{Y}_{11\bullet} \\ \bar{Y}_{13\bullet} - \bar{Y}_{11\bullet} \\ \bar{Y}_{22\bullet} - \bar{Y}_{21\bullet} - \bar{Y}_{12\bullet} + \bar{Y}_{11\bullet} \\ \bar{Y}_{23\bullet} - \bar{Y}_{21\bullet} - \bar{Y}_{13\bullet} + \bar{Y}_{11\bullet} \end{bmatrix} \end{aligned}$$

Restrictions must involve "non-estimable" quantities for the unrestricted "effects" model.

Baseline restrictions: (SAS)

$$\begin{aligned} \alpha_a &= 0 \\ \beta_b &= 0 \\ \gamma_{ib} &= 0 \quad \text{for all } i = 1, \ldots, a \\ \gamma_{aj} &= 0 \quad \text{for all } j = 1, \ldots, b \end{aligned}$$

Baseline restrictions: (S-PLUS)

$$\begin{aligned} \alpha_1 &= 0 \\ \beta_1 &= 0 \\ \gamma_{i1} &= 0 \quad \text{for all } i = 1, \ldots, a \\ \gamma_{1j} &= 0 \quad \text{for all } j = 1, \ldots, b \end{aligned}$$

Σ-restrictions:

$$Y_{ijk} = \underline{\omega + \gamma_i + \delta_j + \eta_{ij}} + \epsilon_{ijk}$$

$\nwarrow$

$$\mu_{ij} = E(Y_{ijk})$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2)$$

$$\sum_{i=1}^{a} \gamma_i = 0$$

$$\sum_{j=1}^{b} \delta_j = 0$$

$$\sum_{i=1}^{a} \eta_{ij} = 0 \quad \text{for each } j = 1, \ldots, b$$

$$\sum_{j=1}^{b} \eta_{ij} = 0 \quad \text{for each } i = 1, \ldots, a$$

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interpretation:

$$\omega = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}$$

is the overall mean germination time, averaging across all soil types and all varieties used in this study.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interpretation:

$$\omega + \delta_j = \bar{\mu}_{.j}$$

$$\delta_j = \bar{\mu}_{.j} - \bar{\mu}_{..}$$

and

$$\delta_j - \delta_k = (\bar{\mu}_{.j} - \bar{\mu}_{..}) - (\bar{\mu}_{.k} - \bar{\mu}_{..})$$
$$= \bar{\mu}_{.j} - \bar{\mu}_{.k}$$

is the difference between mean germination times for varieties j and k, averaging across soil types.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interpretation:

$$\gamma_1 - \gamma_2 = \bar{\mu}_{1.} - \bar{\mu}_{2.}$$

is the difference in the mean germination times for different soil types, averaging across varieties.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interaction:

$$\eta_{ij} = \mu_{ij} - (\omega + \gamma_i + \delta_j)$$

is a deviation from an additive model. Then,

$$\begin{aligned} \eta_{ij} \; - \; & \eta_{kj} - \eta_{i\ell} + \eta_{k\ell} \\ & = \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} \end{aligned}$$

Matrix formulation

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ \gamma_1 \\ \delta_1 \\ \delta_2 \\ \eta_{11} \\ \eta_{12} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{113} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{213} \\ \epsilon_{214} \\ \epsilon_{221} \\ \epsilon_{231} \\ \epsilon_{232} \\ \epsilon_{233} \end{bmatrix}$$

This uses the Σ-restrictions to obtain

$$\begin{array}{ll} \gamma_2 = -\gamma_1 & \delta_3 = -\delta_1 - \delta_2 \\ \eta_{21} = -\eta_{11} & \eta_{13} = -\eta_{11} - \eta_{12} \\ \eta_{22} = -\eta_{12} & \eta_{23} = -\eta_{13} = \eta_{11} + \eta_{12} \end{array}$$

Least squares estimation

$$\text{b} = \begin{bmatrix} \hat{\omega} \\ \hat{\gamma} \\ \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\eta}_{11} \\ \hat{\eta}_{12} \end{bmatrix}$$

$$= (X^T X)^{-1} X^T \text{Y}$$

$$= \begin{bmatrix} \frac{1}{6} \sum_i \sum_j \bar{Y}_{ij.} \\ \frac{1}{3} \sum_j \bar{Y}_{1j.} - \frac{1}{6} \sum \sum \bar{Y}_{ij.} \\ \frac{1}{2} \sum_i \bar{Y}_{i1.} - \frac{1}{6} \sum \sum \bar{Y}_{ij.} \\ \frac{1}{2} \sum_i \bar{Y}_{i2.} - \frac{1}{6} \sum \sum \bar{Y}_{ij.} \\ \bar{Y}_{11.} - \hat{\omega} - \hat{\gamma}_1 - \hat{\delta}_1 \\ \bar{Y}_{12.} - \hat{\omega} - \hat{\gamma}_1 - \hat{\delta}_2 \end{bmatrix} = \begin{bmatrix} 16.83 \\ -3.17 \\ -4.33 \\ 5.67 \\ -0.33 \\ -5.33 \end{bmatrix}$$

If restrictions are placed on "non-estimable"
functions of parameters in the un-restricted
"effects" model, then

- The resulting models are reparameterizations of each other.

- $\hat{Y} = P_X Y$

  $e = Y - \hat{Y} = (I - P_X)Y$

  $SSE = e^T e = Y^T (I - P_X) Y$

  $\hat{Y}^T \hat{Y} = Y^T P_X Y$

  $SS_{\text{model}} = Y^T (P_X - P_1) Y$

  are the same for any set of restrictions.

- The solution to the normal equations

  $$b = (X^T X)^{-1} X^T Y$$

  and interpretations of the corresponding parameters will not be the same for all such sets of restrictions.

If you were to place restrictions on estimable functions of parameters in

$$Y_{ijk} = \mu + \alpha_1 + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

then you would change

- rank(X)

- space spanned by the columns of X

- $\hat{Y} = X(X^T X)^{-} X^T Y$ and OLS estimators of other estimable quantities.

Analysis of variance

$$\begin{aligned} Y^T Y &= Y^T P_\mu Y + Y^T (P_{\mu,\alpha} - P_\mu) Y \\ &\quad + Y^T (P_{\mu,\alpha,\beta} - P_{\mu,\alpha}) Y \\ &\quad + Y^T (P_X - P_{\mu,\alpha,\beta}) Y \\ &\quad + Y^T (I - P_X) Y \\ &= R(\mu) + R(\alpha|\mu) + R(\beta|\mu,\alpha) \\ &\quad + R(\gamma|\mu,\alpha,\beta) + SSE \end{aligned}$$

$$
\begin{bmatrix} Y_{111}\\ Y_{112}\\ Y_{113}\\ Y_{121}\\ Y_{122}\\ Y_{131}\\ Y_{132}\\ Y_{211}\\ Y_{212}\\ Y_{213}\\ Y_{214}\\ Y_{221}\\ Y_{231}\\ Y_{232}\\ Y_{233} \end{bmatrix} =
$$

$$
\left[\begin{array}{c|cc|ccc|cccccc}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} \mu\\ \alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\\ \beta_3\\ \gamma_{11}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{21}\\ \gamma_{22}\\ \gamma_{23} \end{bmatrix} + \epsilon
$$

call this X_μ call this X_α call this X_β call this X_γ

Define:

$X_\mu = X_\mu$ $\qquad P_\mu = X_\mu(X_\mu^T X_\mu)^{-1}X_\mu^T$

$X_{\mu,\alpha} = [X_\mu | X_\alpha]$ $\qquad P_{\mu,\alpha} = X_{\mu,\alpha}(X_{\mu,\alpha}^T X_{\mu,\alpha})^{-}X_{\mu,\alpha}^T$

$X_{\mu,\alpha,\beta} = [X_\mu | X_\alpha | X_\beta]$ $\qquad P_{\mu,\alpha,\beta} = X_{\mu,\alpha,\beta}(X_{\mu,\alpha,\beta}^T X_{\mu,\alpha,\beta})^{-}X_{\mu,\alpha,\beta}^T$

$X = [X_\mu | X_\alpha | X_\beta | X_\gamma]$ $\qquad P_X = X(X^T X)^{-}X^T$

The following three model matrices correspond to reparameterizations of the same model:

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu\\ \alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\\ \beta_3\\ \gamma_{11}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{21}\\ \gamma_{22}\\ \gamma_{23} \end{bmatrix}
$$

$$
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & -1 & -1 & -1 & -1\\
1 & 1 & -1 & -1 & -1 & -1\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 0 & 1 & -1 & 0\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 0 & 1 & 0 & -1\\
1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]
$$

$R(\mu) = Y^T P_\mu Y$ is the same for all three models

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu\\ \alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\\ \beta_3\\ \gamma_{11}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{21}\\ \gamma_{22}\\ \gamma_{23} \end{bmatrix}
$$

$$
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & -1 & -1 & -1 & -1\\
1 & 1 & -1 & -1 & -1 & -1\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 0 & 1 & -1 & 0\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 1 & 0 & -1 & 0\\
1 & -1 & 0 & 1 & 0 & -1\\
1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]
$$

$R(\mu, \alpha) = \mathbf{Y}^T P_{\mu,\alpha} \mathbf{Y}$ is the same for all three models and so is $R(\alpha|\mu) = R(\mu, \alpha) - R(\mu)$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}$$

$$\left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|rrrrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{array}\right]$$

$R(\mu, \alpha, \beta) = \mathbf{Y}^T P_{\mu,\alpha,\beta} \mathbf{Y}$ is the same for all three models and so is $R(\beta|\alpha) = R(\mu, \alpha, \beta) - R(\mu, \alpha)$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}$$

$$\left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|rrrrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{array}\right]$$

$R(\mu, \alpha, \beta, \gamma) = \mathbf{Y}^T P_X \mathbf{Y}$ is the same for all three models and so is $R(\gamma|\mu, \alpha, \beta) = R(\mu, \alpha, \beta, \gamma) - R(\mu, \alpha, \beta)$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}$$

$$\left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|rrrrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{array}\right]$$

Consequently, the partition

$$\begin{aligned} \mathbf{Y}^T\mathbf{Y} &= \mathbf{Y}^T P_\mu \mathbf{Y} + \mathbf{Y}^T (P_{\mu,\beta} - P_\mu)\mathbf{Y} \\ &\quad + \mathbf{Y}^T (P_{\mu,\alpha,\beta} - P_{\mu,\beta})\mathbf{Y} \\ &\quad + \mathbf{Y}^T (P_X - P_{\mu,\alpha,\beta})\mathbf{Y} \\ &\quad + \mathbf{Y}^T (I - P_X)\mathbf{Y} \\ &= R(\mu) + R(\beta|\mu) + R(\alpha|\mu,\beta) \\ &\quad + R(\gamma|\mu,\alpha,\beta) + SSE \end{aligned}$$

is the same for all three models.

Normal Theory Gauss-Markov Model

$$\epsilon_{ijk} \sim NID(0, \sigma^2)$$

By Cochran's Theorem, these quadratic forms (sums of squares) have independent chi-square distributions with $1,\ b-1,\ a-1,$ $(a-1)(b-1)$, and $n_{\bullet\bullet} - ab$ degrees of freedom, respectively, when $n_{ij} > 0$ for all (i,j).

Using result 4.7, we have also shown earlier that

$$\begin{aligned} SSE &= \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} (Y_{ijk} - \bar{Y}_{ij\bullet})^2 \\ &= \mathrm{Y}^T (I - P_X) \mathrm{Y} \\ &\sim \chi^2_{n_{\bullet\bullet} - ab} \end{aligned}$$

The $\ell m(\ )$ function in S-PLUS:

To allow the $\ell m(\ )$ function to fit a model involving classification variables, create factors.

```
> carrot ← read.table("carrots.dat",
      col.names=c("Soil","Variety","Days"))
> carrot$soil ← as.factor(carrot$soil)
> carrot$variety ←
      as.factor(carrot$variety)
> options(contrasts=
      c("contr.sum","contr.poly"))
```


Produce ANOVA tables:

```
> ℓm.out1 ← ℓm(Days ~ soil*variety,
                data=carrot)
> anova (ℓm.out1)
```

$R(\alpha|\mu)$
$R(\beta|\mu,\alpha)$
$R(\gamma|\mu,\alpha,\beta)$
SSE

```
> ℓm.out2 ← ℓm(Days ~ variety*soil,
                data=carrot)
> anova (ℓm.out2)
```

$R(\beta|\mu)$
$R(\alpha|\mu,\beta)$
$R(\gamma|\mu,\alpha,\beta)$
SSE

There are four options for creating columns in the model matrix for classification variables:

contr. helmert	
contr. treatment	sets $\alpha_1 = 0$ $\beta_1 = 0$ $\gamma_{1j} = 0$ for all j $\gamma_{i1} = 0$ for all i
contr.sum	$\Sigma -$ constraints
contr.poly	orthogonal polynomial contrasts • equal spacing • equal sample sizes

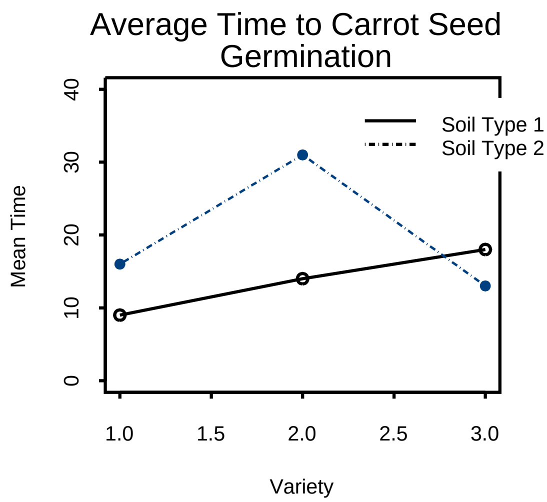

```
># This file is posted as  carrots.ssc

> carrot <- read.table("c:\\carrots.dat",
+    col.names=c("Soil","Variety","Days"))
> carrot$Soil <- as.factor(carrot$Soil)
> carrot$Variety <- as.factor(carrot$Variety)
> carrot

   Soil Variety Days
 1    1       1    6
 2    1       1   10
 3    1       1   11
 4    1       2   13
 5    1       2   15
 6    1       3   14
 7    1       3   22
 8    2       1   12
 9    2       1   15
10    2       1   19
11    2       1   18
12    2       2   31
13    2       3   18
14    2       3    9
15    2       3   12
```



```
# Compute sample means of germination
# times for all combinations of soil
# type and varieties of carrot seeds
# and make a profile plot.

> means <- tapply(carrot$Days,
   list(carrot$Variety,carrot$Soil),mean)
> means

   1  2
1  9 16
2 14 31
3 18 13
```



```
# Set up the axes and title of the
# profile plot.

> par(fin=c(7,7),cex=1.2,lwd=3,mex=1.5)
> x.axis <- unique(carrot$Variety)
> matplot(c(1,3,1), c(0,40,10), type="n",
     xlab="Variety",  ylab="Mean Time",
     main= "Average Time to Carrot Seed
Germination")

#  Add a profile for each soil type

> matlines(x.axis,means,type='l',
            lty=c(1,3),lwd=3)

# Plot points for the observations

> matpoints(x.axis,means, pch=c(1,16))

# Add a legend to the plot

> legend(2.,38.6,
   legend=c('Soil Type 1','Soil Type 2'),
             lty=c(1,3),bty='n')
```




```
# Fit a model with main effects and interaction
# Compute both sets of Type I sums of squares

> options(contrasts=
      c(''contr.sum'',''contr.poly''))
> lm.out1 <- lm(Days~Soil*Variety,data=carrot)
> anova(lm.out1)


Analysis of Variance Table

Response: Days
Terms added sequentially (first to last)
             Df   Sum Sq  Mean Sq  F Value  Pr(F)
        Soil  1   52.500  52.5000 3.937500 0.0785
     Variety  2  124.734  62.3670 4.677527 0.0405
Soil:Variety  2  222.766 111.3830 8.353723 0.0089
   Residuals  9  120.000  13.3333
```

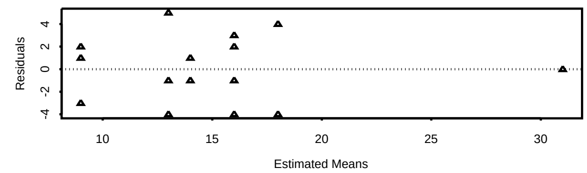


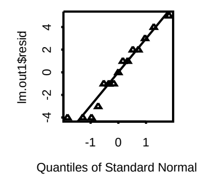
```
> lm.out2 <- lm(Days~Variety*Soil,data=carrot)
> anova(lm.out2)

Analysis of Variance Table

Response: Days
Terms added sequentially (first to last)
             Df   Sum Sq  Mean Sq  F Value  Pr(F)
     Variety  2  93.3333  46.6667 3.500000 0.0751
        Soil  1  83.9007  83.9007 6.292553 0.0334
Variety:Soil  2 222.7660 111.3830 8.353723 0.0089
   Residuals  9 120.0000  13.3333
```



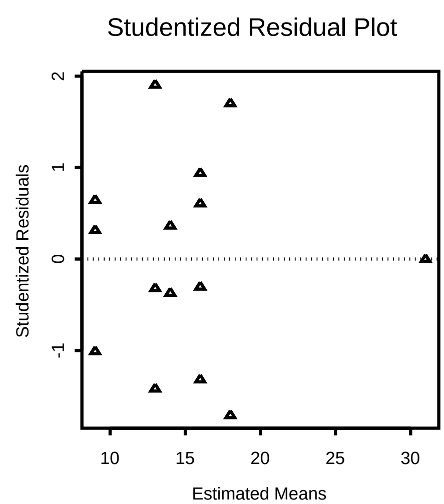
```
# Create a data frame containing the original
# data and the residuals and estimated means

> data.frame(carrot$Soil,carrot$Variety,
              carrot$Days,
              Pred=lm.out1$fitted,
             Resid=round(lm.out1$resid,3))

   X1 X2 X3 Pred Resid
 1  1  1  6    9    -3
 2  1  1 10    9     1
 3  1  1 11    9     2
 4  1  2 13   14    -1
 5  1  2 15   14     1
 6  1  3 14   18    -4
 7  1  3 22   18     4
 8  2  1 12   16    -4
 9  2  1 15   16    -1
10  2  1 19   16     3
11  2  1 18   16     2
12  2  2 31   31     0
13  2  3 18   13     5
14  2  3  9   13    -4
15  2  3 12   13    -1
```

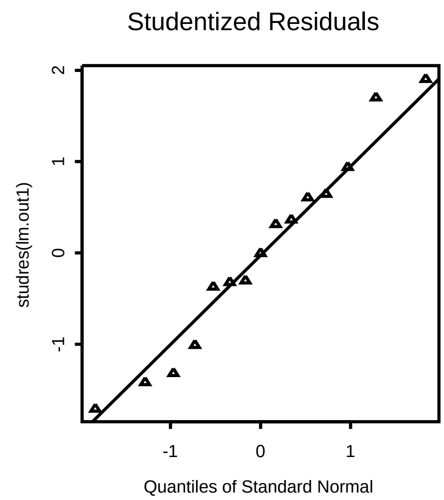
```
# Create residual plots

frame( )
par(cex=1.0,mex=1.0,lwd=3,pch=2,
     mkh=0.1,fig=c(0,1,.51,1), pty='m')
  plot(lm.out1$fitted, lm.out1$resid,
       xlab="Estimated Means",
       ylab="Residuals")
  abline(h=0, lty=2, lwd=3)

par(fig=c(0, 1, 0, 0.49), pty='s')
  qqnorm(lm.out1$resid)
  qqline(lm.out1$resid)
```




```
# Create plots for studentized residuals
# You must attach the MASS library
# to have access to the studres( )
# function that computes studentized
# residuals in the following code

library(MASS)

frame( )
par(cex=1.0,mex=1.0,lwd=3,pch=2,
     mkh=0.1,fin=c(6.5,6.5))
  plot(lm.out1$fitted, studres(lm.out1),
       xlab="Estimated Means",
       ylab="Studentized Residuals",
       main="Studentized Residual Plot")
  abline(h=0, lty=2, lwd=3)

  qqnorm(studres(lm.out1),
       main="Studentized Residuals")
  qqline(studres(lm.out1))
```




```
# Compute Type III sums of squares and F-tests.
# First create the model matrix for
# the cell means model.

> cb <- as.factor(10*as.numeric(carrot$Soil)
          + as.numeric(carrot$Variety))
> lm.out <- lm(carrot$Days ~ cb - 1)
> D <- model.matrix(lm.out)
> D
```

	cb11	cb12	cb13	cb21	cb22	cb23
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	0	1	0	0	0	0
5	0	1	0	0	0	0
6	0	0	1	0	0	0
7	0	0	1	0	0	0
8	0	0	0	1	0	0
9	0	0	0	1	0	0
10	0	0	0	1	0	0
11	0	0	0	1	0	0
12	0	0	0	0	1	0
13	0	0	0	0	0	1
14	0	0	0	0	0	1
15	0	0	0	0	0	1


```
# Compute the sample means

> y <- matrix(carrot$Days,ncol=1)
> b <- solve(crossprod(D)) %*% crossprod(D,y)
> b

     [,1]
cb11    9
cb12   14
cb13   18
cb21   16
cb22   31
cb23   13


# Generate an identity matrix and
#    a vector of ones

Iden <- function(n) diag(rep(1,n))
one  <- function(n) matrix(rep(1,n),ncol=1)
```



```
# Compute Type III sums of squares and
# related F-tests

> s  <- length(unique(carrot$Soil))
> t  <- length(unique(carrot$Variety))

> yhat <- D %*% b
> sse  <- crossprod(y-yhat)
> df2  <- nrow(y) - s*t

> c1 <- kronecker( cbind(Iden(s-1),-one(s-1)),
                 t(one(t)) )
> q1 <- t(b) %*% t(c1)%*%
          solve( c1 %*% solve(crossprod(D))
           %*% t(c1))%*% c1 %*% b
> df1<- s-1
> f <- (q1/df1)/(sse/df2)
> p <- 1-pf(f,df1,df2)
```

```
> c1

     [,1] [,2] [,3] [,4] [,5] [,6]
[1,]    1    1    1   -1   -1   -1


> data.frame(SS=q1,df=df1,F.stat=f,p.value=p)

        SS df   F.stat    p.value
1 123.7714  1 9.282857 0.01386499


> c2 <- kronecker( t(one(s)), cbind(Iden(t-1),
           -one(t-1)))
> q2 <- t(b) %*% t(c2)%*%
         solve( c2 %*% solve(crossprod(D))
            %*% t(c2))%*% c2 %*% b
> df1<- t-1
> f  <- (q2/df1)/(sse/df2)
> p  <- 1-pf(f,df1,df2)
> c2

     [,1] [,2] [,3] [,4] [,5] [,6]
[1,]    1    0   -1    1    0   -1
[2,]    0    1   -1    0    1   -1
```



```
> data.frame(SS=q2,df=df1,F.stat=f,p.value=p)

        SS  df    F.stat     p.value
1 192.1277   2  7.204787  0.01354629

> c3 <- kronecker( cbind(Iden(s-1),-one(s-1)),
                cbind(Iden(t-1),-one(t-1)) )
> q3 <- t(b) %*% t(c3)%*%
            solve( c3 %*% solve(crossprod(D))
               %*% t(c3))%*% c3 %*% b
> df1<- (s-1)*(t-1)
> f  <- (q3/df1)/(sse/df2)
> p  <- 1-pf(f,df1,df2)
> c3

     [,1] [,2] [,3] [,4] [,5] [,6]
[1,]    1    0   -1   -1    0    1
[2,]    0    1   -1    0   -1    1

> data.frame(SS=q3,df=df1,F.stat=f,p.value=p)

       SS  df    F.stat     p.value
1 222.766   2  8.353723  0.00888845
```


What null hypotheses are tested by F-tests derived from such ANOVA tables?

Consider Type I sums of squares:

$$\begin{aligned} R(\mu) &= \mathbf{Y}^T P_1 \mathbf{Y} \\ &= \mathbf{Y}^T P_1 P_1 \mathbf{Y} \\ &= (P_1\mathbf{Y})^T (P_1 \mathbf{Y}) \\ &= (\bar{Y}_{...}1)^T(\bar{Y}_{...}1) = n_{..}\bar{Y}^2_{...} \end{aligned}$$

$\frac{1}{\sigma^2}R(\mu) \sim \chi^2_1(\delta^2)$ and

$$F = \frac{R(\mu)}{SSE/(n_{..}-ab)} \sim F_{(1,n_{..}-ab)}(\delta^2)$$

where

$$\begin{aligned} \delta^2 &= \frac{1}{\sigma^2}\, \beta^T X^T P_1 X\beta \\ &= \frac{1}{\sigma^2}\, (\beta^T X^T P_1)(P_1 X\beta) \\ &= \frac{1}{\sigma^2}\, (P_1 X\beta)^T (P_1 X\beta) \end{aligned}$$

For the carrot seed germination study:

$$\begin{aligned} P_1 X\beta &= \frac{1}{n_{..}} 1\, 1^T X\beta \\ &= \frac{1}{n_{..}}\, 1 [n_{..}, n_{1.}, n_{2.}, n_{.1}, n_{.2}, n_{.3}, \\ &\quad n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}]\ \beta \\ &= \frac{1}{n_{..}}\ 1\ \Big(n_{..}\mu + \sum_{i=1}^a n_{i.}\alpha_i + \sum_{j=1}^b n_{.j}\beta_j \\ &\quad + \sum_{i=1}^a \sum_{j=1}^b \gamma_{ij}\Big) \end{aligned}$$

The null hypothesis is

$$H_0: 0 = n_{..}\,\mu + \sum_{i=1}^a n_{i.}\,\alpha_i + \sum_{j=1}^b n_{.j}\,\beta_j + \sum_i \sum_j n_{ij}\gamma_{ij}$$

With respect to the cell means

$$E(Y_{ijk}) = \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

this null hypothesis is

$$H_0: 0 = \sum_{i=1}^a \sum_{j=1}^b \frac{n_{ij}}{n_{..}}\,\mu_{ij}$$

Consider

$$R(\alpha|\mu) = \mathrm{Y}^T(P_{\mu,\alpha} - P_\mu)\mathrm{Y}$$

and

$$F = \frac{R(\alpha|\mu)/(a-1)}{MSE} \sim F_{(a-1, n_{..}-ab)}(\delta^2)$$

Here,

$$\frac{1}{\sigma^2} R(\alpha|\mu) \sim \chi^2_{a-1}(\delta^2)$$

where $a-1 =$ rank$(X_{\mu,\alpha})-$ rank(X_μ) and

$$\begin{aligned} \delta^2 &= \frac{1}{\sigma^2}\beta^T X^T (P_{\mu,\alpha} - P_\mu) X\beta \\ &= \frac{1}{\sigma^2}\left[(P_{\mu,\alpha} - P_\mu)X\beta\right]^T \left[(P_{\mu,\alpha} - P_\mu)X\beta\right] \end{aligned}$$

For the general effects model for the carrot seed germination study:

$$\begin{aligned} P_{\mu,\alpha} X &= X_{\mu,\alpha}(X^T_{\mu,\alpha} X_{\mu,\alpha})^- X^T_{\mu,\alpha} X \\ &= X_{\mu,\alpha} \begin{bmatrix} n_{..} & n_{1.} & n_{2.} \\ n_{1.} & n_{1.} & 0 \\ n_{2.} & 0 & n_{2.} \end{bmatrix}^- \end{aligned}$$

$$\begin{bmatrix} n_{..} & n_{1.} & n_{2.} & n_{.1} & n_{.2} & n_{.3} & n_{11} & n_{12} & n_{13} & n_{21} & n_{22} & n_{23} \\ n_{1.} & n_{1.} & 0 & n_{11} & n_{12} & n_{13} & n_{11} & n_{12} & n_{13} & 0 & 0 & 0 \\ n_{2.} & 0 & n_{2.} & n_{21} & n_{22} & n_{23} & 0 & 0 & 0 & n_{21} & n_{22} & n_{23} \end{bmatrix}$$

$$\downarrow$$

$$= X_{\mu,\alpha} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{n_{1.}} & 0 \\ 0 & 0 & \frac{1}{n_{2.}} \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{n_{11}}{n_{1.}} & \frac{n_{12}}{n_{1.}} & \frac{n_{13}}{n_{1.}} & \frac{n_{11}}{n_{..}} & \frac{n_{12}}{n_{..}} & \frac{n_{13}}{n_{..}} & 0 & 0 & 0 \\ 1 & 0 & 1 & \frac{n_{11}}{n_{1.}} & \frac{n_{22}}{n_{2.}} & \frac{n_{23}}{n_{2.}} & 0 & 0 & 0 & \frac{n_{21}}{n_{1.}} & \frac{n_{22}}{n_{2.}} & \frac{n_{23}}{n_{1.}} \end{bmatrix}$$

Then, the first seven rows of $\underline{(P_{\mu,\alpha} - P_\mu)X\beta}$ are

$$\begin{aligned} &[\mu + \alpha_1 + \textstyle\sum_{j=1}^b \frac{n_{1j}}{n_{1.}}\ (\beta_j + \gamma_{1j})] \\ &\qquad -[\mu + \sum_{i=1}^a \frac{n_{i.}}{n_{..}}\alpha_i + \sum_{j=1}^b \frac{n_{.j}}{n_{..}}\beta_j + \sum_i\sum_j \frac{n_{ij}}{n_{..}}\gamma_{ij}] \end{aligned}$$

The last eight rows of $(P_{\mu,\alpha} - P_\mu)X\beta$ are

$$\begin{aligned} &[\mu + \alpha_2 + \textstyle\sum_{j=1}^b \frac{n_{2j}}{n_{2.}}\ (\beta_j + \gamma_{2j})] \\ &\qquad -[\mu + \sum_{i=1}^a \frac{n_{i.}}{n_{..}}\alpha_i + \sum_{j=1}^b \frac{n_{.j}}{n_{..}}\beta_j + \sum_i\sum_j \frac{n_{ij}}{n_{..}}\gamma_{ij}] \end{aligned}$$

The null hypothesis is

$$H_0 : \alpha_i + \sum_{j=1}^b \frac{n_{ij}}{n_{i.}}(\beta_j + \gamma_{ij})$$

are all equal $(i = 1, \ldots, a)$

with respect to the cell means model,

$$\mu_{ij} = E(Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij}\ ,$$

this null hypothesis is

$H_0 : \sum_{j=1}^b \frac{n_{ij}}{n_{i.}}\ \mu_{ij}$ are all equal $(i = 1, \ldots, a)$.