

Reading Assignment: Finish your review of Chapter 2 in Rencher's book. Then read Chapter 3 and begin to read Chapter 11.

Written Assignment: On-campus students: Due Wednesday, January 30, in class.  
Distance students: Send it by Thursday, February 7.

The first three problems on this assignment ask you to provide mathematical proofs of some matrix properties. Problems 4 and 6 ask you to use S-PLUS to numerically evaluate some quantities. Problem 5 asks you to write an S-PLUS function to convert a covariance matrix into a correlation matrix. In problem 7, you are asked to use formulas for the inverse of a matrix and numerical evaluation of inverses of larger matrices to find a formula for the inverse of a special type of covariance matrix.

1. Use the definition of orthogonal and idempotent matrices and properties of determinants in the lecture notes to prove the following results:
  - (a) If  $A$  is an orthogonal matrix, then  $|A|$  is either 1 or -1. (Hint: use the definition of an orthogonal matrix and consider the determinant of an identity matrix.)
  - (b) If  $W$  is an idempotent matrix, then  $|W|$  is either 0 or 1.
2. Let  $A$  be a  $p \times p$  positive definite matrix and let  $B$  be a  $k \times p$  matrix of rank  $k \leq p$ . Show that  $BAB^T$  is positive definite. (This is Corollary 1 on page 23 of Rencher's book.)
3. Show that a symmetric matrix  $A$  is positive definite if and only if there exists a nonsingular matrix  $P$  such that  $A = P^T P$ . (This is Theorem 2.6C on page 23 of Rencher's book.)
4. Consider the matrix

$$V = \begin{bmatrix} 5.0 & 4.0 & 3.2 \\ 4.0 & 5.0 & 4.0 \\ 3.2 & 4.0 & 5.0 \end{bmatrix}.$$

Use S-PLUS to find values for

- (a) the eigenvalues and eigenvectors of  $V$
- (b) trace ( $V$ )
- (c)  $\det(V)$
- (d) the inverse of  $V$

5. Suppose  $\tilde{Y} = (Y_1 \ Y_2 \ \dots Y_n)^T$  is a random vector with mean vector  $E(\tilde{Y}) = \tilde{\mathbf{m}}$  and covariance matrix  $\text{Var}(\tilde{Y}) = \tilde{\Sigma}$ .

- Describe a matrix  $\mathbf{B}$  such that  $\mathbf{B}\tilde{\Sigma}\mathbf{B}$  is the correlation matrix for  $\tilde{Y}$ .
- Write an S-PLUS function to compute the correlation matrix corresponding to any covariance matrix. Submit a listing of your function.
- Use the S-PLUS function you created in part (b) of this problem to compute the correlation matrix when  $\text{Var}(\tilde{Y}) = \mathbf{V}$ , the matrix from problem 4.

6. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}.$$

These matrices are nearly identical. There is only a small difference in the (2,2) elements of these matrices. Moreover, the columns of  $\mathbf{A}$  (and the columns of  $\mathbf{B}$ ) are nearly linearly dependent. Use S-PLUS to compute the following quantities:

- The determinant of  $\mathbf{A}$  and the determinant of  $\mathbf{B}$ .
  - $\mathbf{A}^{-1}$  and  $\mathbf{B}^{-1}$ . (Note that  $\mathbf{A}^{-1}$  is approximately  $-3\mathbf{B}^{-1}$ , even though  $\mathbf{A}$  is nearly identical to  $\mathbf{B}$ . This shows that small changes in elements of matrices that are nearly singular, can have big effects on some matrix operations. Consequently, failure to retain enough significant digits can result in substantial computational errors.)
7. Software such as S-PLUS, MATLAB or the IML procedure in SAS can be useful in the development of statistical theory and mathematical results. You should take advantage of it in your theory courses and your research. If you are asked to prove a result and you are not sure if it is true, for example, use S-PLUS to evaluate a few numerical examples. You only need to find one example where the result does not hold to show it is false. If the result holds true for every numerical example that you try, you may begin to believe that it is indeed true and try to develop a proof. Computational tools can also be useful in discovering patterns.

Here we will use S-PLUS to help derive a formula for the inverse of an  $n \times n$  matrix of the following form:

$$V = \sigma^2 \begin{pmatrix} 1 & b & b^2 & b^3 & \dots & b^{n-1} \\ b & 1 & b & b^2 & \dots & b^{n-2} \\ b^2 & b & 1 & b & \dots & b^{n-3} \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ b^{n-2} & b^{n-3} & \dots & b & 1 & b \\ b^{n-1} & b^{n-2} & \dots & b^2 & b & 1 \end{pmatrix}$$

Covariance matrices corresponding to a first order auto-correlation structure have this form. In that case,  $b$  would be a first order auto-correlation. You are first asked to derive a formula for the  $2 \times 2$  and  $3 \times 3$  cases. Next you are asked to use S-PLUS to evaluate the inverse for several special cases and then infer the general result. You may have to evaluate some additional examples to recognize the pattern in the results. Once a formula for  $V^{-1}$  is identified, you can easily prove that it is the inverse of  $V$  by directly checking that  $V^{-1}V = VV^{-1} = I$ . Begin your investigation by finding a formula for the inverse of the correlation matrix  $R$  associated with  $V$ .

(a) First use Result 1.4 (i) in the lecture notes to obtain the formula for the inverse of

$$R = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

(b) Now use Result 1.4 (ii) in the lecture notes to obtain the formula for the inverse of

$$R = \begin{pmatrix} 1 & b & b^2 \\ b & 1 & b \\ b^2 & b & 1 \end{pmatrix}$$

(c) Now use S-PLUS to evaluate the inverse of

$$R = \begin{pmatrix} 1 & b & b^2 & b^3 \\ b & 1 & b & b^2 \\ b^2 & b & 1 & b \\ b^3 & b^2 & b & 1 \end{pmatrix}$$

for  $b=0.1$  and  $b=0.5$ . Evaluate the inverse of the  $5 \times 5$  version of  $R$  for  $b=0.1$  and  $b=0.5$ . You may need to evaluate some additional cases with other values of  $b$  to see the pattern? Do not report numerical results for the examples, but use the numerical results and the results from parts (a) and (b) to guess a formula for the inverse

of  $R$ . Use S-PLUS to numerically verify that your guess is correct. When you are satisfied with your solution to the inverse for  $R$ , use it to obtain a formula for the inverse of  $V$ .

If you are unfamiliar with matrix algebra, you may wish to work some of the problems at the end of Chapter 2 in Rencher's book. (Do not submit answers to these problems. They are optional.) You could work some of the numerical problems without the aid of a calculator or computer and then check your answers by obtaining the solutions with S-PLUS. Some of these problems include 2.3, 2.4, 2.14, 2.15, 2.17, 2.30, 2.53, 2.56, 2.59, 2.60, 2.70, 2.72, 2.74, 2.76, 2.79. Other problems that deal more with concepts and definitions are 2.22, 2.23, 2.27, 2.28, 2.80, 2.29, 2.34, 2.35, 2.57, 2.68, 2.69, 2.71. Some of the other problems at the end of Chapter 2 deal with generalized inverses of matrices which we will address on future assignments.