9.1

\[
L' = [0.9 \quad 0.7 \quad 0.5]; \quad LL' = \begin{bmatrix}
0.81 & 0.63 & 0.45 \\
0.63 & 0.49 & 0.35 \\
0.45 & 0.35 & 0.25
\end{bmatrix}
\]

so \( \mathbf{p} = LL' + \mathbf{y} \)

9.2 a) For \( m = 1 \)
\[
\begin{align*}
h_1^2 = r_{11}^2 &= 0.81 \\
h_2^2 = r_{21}^2 &= 0.49 \\
h_3^2 = r_{31}^2 &= 0.25
\end{align*}
\]

The communalities are those parts of the variances of the variables explained by the single factor.
b) Corr($Z_i, F_1$) = Cov($Z_i, F_1$), $i = 1, 2, 3$. By (9-5) Cov($Z_i, F_1$) = $\xi_{i1}$.

Thus Corr($Z_1, F_1$) = $\xi_{11}$ = .9; Corr($Z_2, F_1$) = $\xi_{21}$ = .7; Corr($Z_3, F_1$) = $\xi_{31}$ = .5. The first variable, $Z_1$, has the largest correlation with the factor and therefore will probably carry the most weight in naming the factor.

9.3

a) $L = \sqrt{\lambda_1} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = \sqrt{1.95} \begin{bmatrix} .625 \\ .593 \end{bmatrix} = \begin{bmatrix} .876 \\ .831 \end{bmatrix}$. Slightly different from result in Exercise 9.1.

b) Proportion of total variance explained = $\frac{\lambda_1}{p} = \frac{1.95}{3} = .65$

9.4

$\Psi = \Psi - \Psi = LL'$ = $\begin{bmatrix} .81 & .63 & .45 \\ .63 & .49 & .35 \\ .45 & .35 & .25 \end{bmatrix}$

$L = \sqrt{\lambda_1} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = \sqrt{1.55} \begin{bmatrix} .7229 \\ .5623 \\ .4016 \end{bmatrix} = \begin{bmatrix} .9 \\ .7 \\ .5 \end{bmatrix}$

Result is consistent with results in Exercise 9.1. It should be since $m = 1$ common factor completely determines $\hat{\Psi} = \Psi - \Psi$.

9.7

From the equation $\Sigma = LL' + \Psi$, $m = 1$, we have

$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \xi_{11}^2 + \Psi_1 & \xi_{11} \xi_{21} \\ \xi_{11} \xi_{21} & \xi_{21}^2 + \Psi_2 \end{bmatrix}$

so $\sigma_{11} = \xi_{11}^2 + \Psi_1$, $\sigma_{22} = \xi_{21}^2 + \Psi_2$ and $\sigma_{12} = \xi_{11} \xi_{21}$.

Let $\rho = \sigma_{12} / \sqrt{\sigma_{11} \sigma_{22}}$. Then, for any choice $|\rho| \sqrt{\sigma_{22}} \leq \xi_{21}$, $\leq \sqrt{\sigma_{22}}$, set $\xi_{11} = \sigma_{12} / \xi_{21}$ and check $\sigma_{12} = \xi_{11} \xi_{21}$. We obtain $\Psi_1 = \sigma_{11} - \xi_{11}^2 = \sigma_{11} - \frac{\sigma_{12}^2}{\xi_{21}^2} \geq \sigma_{11} - \frac{\sigma_{12}^2}{\rho^2 \sigma_{22}} = \sigma_{11} - \sigma_{11} = 0$ and $\Psi_2 = \sigma_{22} - \xi_{21}^2 \geq \sigma_{22} - \sigma_{22} = 0$. Since $\xi_{21}$ was arbitrary within a suitable interval, there are an infinite number of solutions to the factorization.
\[ \Sigma = LL' + \Psi \quad \text{for } m = 1 \text{ implies} \]

\[
\begin{bmatrix}
1 = \xi_{11}^2 + \psi_1 & .4 = \xi_{11} \xi_{21} & .9 = \xi_{11} \xi_{31} \\
1 = \xi_{21}^2 + \psi_2 & .7 = \xi_{21} \xi_{31} \\
& 1 = \xi_{31}^2 + \psi_3
\end{bmatrix}
\]

Now \( \xi_{11} = \frac{.9}{.7} \) and \( \xi_{11} \xi_{21} = .4 \), so \( \xi_{11}^2 = \left(\frac{.9}{.7}\right)(.4) \) and \( \xi_{11} = \pm .717 \). Thus \( \xi_{21} = \pm .558 \). Finally, from \( .9 = \xi_{11} \xi_{31} \), we have \( \xi_{31} = \pm .9 / .717 = \pm 1.255 \).

Note all the loadings must be of the same sign because all the covariances are positive. We have

\[
LL' = \begin{bmatrix}
.717 \\
.558 \\
1.255
\end{bmatrix}
\begin{bmatrix}
.717 & .558 & 1.255
\end{bmatrix} = \begin{bmatrix}
.514 & .4 & .9 \\
.4 & .3111 & .7 \\
.9 & .7 & 1.575
\end{bmatrix}
\]

so \( \psi_3 = 1 - 1.575 = -.575 \), which is inadmissible as a variance.

\[ \sqrt{9.9} \]

(a) Stoeltzel's interpretation seems reasonable. The first factor seems to contrast sweet with strong liquors.

(b)

\[ \text{Factor 2} \]

1.0

- - -

Marc

.5

- - -

Calvados

- - -

Cognac

- - -

Whiskey

- - -

Armagoc

- - -

Liquors

5

- - -

Kirsch

1.0

- - -

Mirabelle

- - -

It doesn't appear as if rotation of the factor axes is necessary.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Specific Variance</th>
<th>Communality</th>
</tr>
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<tbody>
<tr>
<td>Skull length</td>
<td>.5975</td>
<td>.4024</td>
</tr>
<tr>
<td>Skull breadth</td>
<td>.7582</td>
<td>.2418</td>
</tr>
<tr>
<td>Femur length</td>
<td>.1221</td>
<td>.8779</td>
</tr>
<tr>
<td>Tibia length</td>
<td>.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Humerus length</td>
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<td>.9905</td>
</tr>
<tr>
<td>Ulna length</td>
<td>.0938</td>
<td>.9052</td>
</tr>
</tbody>
</table>

(c) The proportion of variance explained by each factor is:

Factor 1: \[ \frac{1}{6} \sum_{i=1}^{6} t_{1i}^2 = \frac{4.0001}{6} \text{ or } 66.7\% \]

Factor 2: \[ \frac{1}{6} \sum_{i=1}^{6} t_{2i}^2 = \frac{.4177}{6} \text{ or } 6.7\% \]

(d) \[ R - \hat{\psi} = \begin{bmatrix} 0 & .193 & 0 \\ -.017 & -.032 & 0 \\ .000 & .000 & .000 & 0 \\ -.000 & .001 & .000 & .000 & 0 \\ -.001 & -.018 & .003 & .000 & .000 & 0 \end{bmatrix} \]